Getting Started With Nonstandard Methods – Progress Report

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Abstract

An important application of logic to mathematics is the development of nonstandard analysis. We study some concepts in this important area and show how to prove various results using nonstandard methods. This progress report is a part of a larger project involving nonstandard methods.

The superstructure is a fundamental construct in nonstandard analysis. A superstructure $V(X)$ is constructed from some set X by recursively taking power sets. The starting "level" of a superstructure, denoted by $V_0(X)$, is simply the original set X. Any subsequent level, $V_n(X)$, is simply the union of $V_{n-1}(X)$ and the power set of $V_{n-1}(X)$. An element is said to have rank n if it is in $V_n(X)$ but not in $V_{n-1}(X)$.

Superstructures are interesting in that they can be shown to contain any given mathematical entity related to the original set. A formal language is used to formulate sentences that describe such mathematical entities and their properties. The transfer principle dictates that any nonstandard results correspond naturally to standard results, and vice versa.

We demonstrate how an informal sentence is translated into a formal sentence. We also compute the rank of several familiar entities from linear algebra, such as vector spaces, linear transformations, homomorphisms, unordered bases, and ordered bases. We then prove a fundamental theorem about monomorphisms, and proceed to show how vector spaces fit into the nonstandard framework.

Superstructures

The background material for our study of superstructures is from [1].

Definition. Let X be a nonempty set containing at least the natural numbers N. The *power set* $\mathcal{P}(X)$ of X is the set of all subsets of X (including the empty set $\emptyset$). The nth *cumulative power set* $V_n(X)$ of X is defined recursively by

$$V_0(X) = X, \qquad V_{n+1}(X) = V_n(X) \cup \mathcal{P}(V_n(X)).$$

The *superstructure* over X is the set

$$V(X) = \bigcup_{n=0}^{\infty} V_n(X).$$

The elements of $V(X)$ are called *entities*, and the entries in X are also called *individuals*. The entities in $V_n(X) - V_{n-1}(X)$ are of rank n.

The following figure is an illustration of a superstructure. The set $\{\{x\},\{x,y\}\}$ is the ordered pair (x,y).

Lemma 1. Let $V(X)$ be a superstructure, let $a, b \in V(X)$, and let $b \in a$. Also, suppose that b has the maximum rank of any element of a. Then, $rank(a) = rank(b) + 1$.

Proof. Suppose b has rank n. Then, $b \in V_n(X)$. Because a has no elements with a rank higher than n, $a \subseteq V_n(X)$. Since $b \in V_n(X) - V_{n-1}(X)$, and $b \in a$, $a \notin V_n(X)$. It follows that $a \in V_{n+1}(X)$ and therefore $rank(a) = n + 1 = rank(b) + 1$. □

Lemma 2. Let $V(X)$ be a superstructure, and let $x, y \in V_0(X)$. Then, the ordered pair (x, y) is in $V(X)$ and has rank 2.
The figure above shows one way to find the rank of an ordered pair.

Sentences in a formal language can be written to describe any entity in a superstructure.

Example. The interpretation of the following sentence is the informal statement "$f : A \to B$ is a function":

$$(\forall x \in A)(\forall y \in B)(\forall z \in B)[((x,y) \in f \wedge (x,z) \in f) \Rightarrow y = z].$$

The interpretation of the following sentence is "$f : A \to B$ is onto":

$$(\forall z \in A)(\exists x \in B)[(x,z) \in f].$$

Vector Spaces

Our notation for vector spaces is derived from the notation used in [2] for modules over a ring with identity.

Definition. Let $\mathbb{F}$ be a field. A *vector space* is an algebra $(V, +, -, 0, (f_r)_{r\in\mathbb{F}})$ where $+$ is binary, $-$ is unary, 0 is nullary, and each f_r is unary, such that the following identities hold:

(i) $x + (y + z) = (x + y) + z$

(ii) $x + 0 = 0 + x = x$

(iii) $x + (-x) = (-x) + x = 0$

(iv) $x + y = y + x$

(v) $f_r(x + y) = f_r(x) + f_r(y)$, for all $r \in F$

(vi) $f_{r+s}(x) = f_r(x) + f_s(x)$, for all $r, s \in F$

(vii) $f_r(f_s(x)) = f_{rs}(x)$, for all $r, s \in F$

(viii) $f_1(x) = x$

From a universal algebra standpoint, the important functions between vector spaces are homomorphisms.

Definition. Let $\mathbb{V} = (V, +^{\mathbb{V}}, -^{\mathbb{V}}, 0^{\mathbb{V}}, (f_r)^{\mathbb{V}}_{r\in\mathbb{F}})$, $\mathbb{W} = (V, +^{\mathbb{W}}, -^{\mathbb{W}}, 0^{\mathbb{W}}, (f_r)^{\mathbb{W}}_{r\in\mathbb{F}})$. Then, $h : V \to W$ is a *vector space homomorphism* if it satisfies the following properties:

(i) $(\forall x, y \in V)(h(x +^{\mathbb{V}} y) = h(x) +^{\mathbb{W}} h(y))$

(ii) $(\forall x \in V)(h(-^{\mathbb{V}}x) = -^{\mathbb{W}}h(x))$

(iii) $h(0^{\mathbb{V}}) = 0^{\mathbb{W}}$

(iv) For each $r \in \mathbb{F}$, $(\forall x \in V)(h(f_r^{\mathbb{V}}(x)) = f_r^{\mathbb{W}}(h(x)))$

The classical treatment of vector spaces relies on the study of linear transformations between vector spaces.

Definition. $L : V \to V$ is a *linear transformation* if it satisfies:

(i) $(\forall x, y \in V)(L(x +^{\mathbb{V}} y) = L(x) +^{\mathbb{W}} L(y))$,

(ii) For each $r \in F$, $(\forall x \in V)(L(f_r^{\mathbb{V}}(x)) = f_r^{\mathbb{W}}(L(x)))$.

Proposition. A vector space in a superstructure has rank 9.

The following diagram illustrates how the rank of the $-$ operation is determined.

Proposition. A vector space homomorphism in a superstructure has rank 3.

Proposition. Every linear transformation is a homomorphism and vice versa.

Unordered bases are studied in classical vector space theory. The definition of a basis requires the concept of linear independence.

Definition. Let $\mathcal{F} = \{x_1, \ldots, x_n\} \subseteq V$, with $|\mathcal{F}| = n$. Then $\mathcal{F}$ is *linearly dependent* if there exist scalars $r_1, \ldots, r_n \in F$, not all zero, such that

$$f_{r_1}^{\mathbb{V}}(x_1) +^{\mathbb{V}} \cdots +^{\mathbb{V}} f_{r_n}^{\mathbb{V}}(x_n) = 0^{\mathbb{V}}.$$

Otherwise, $\mathcal{F}$ is *linearly independent*.
The concept of a spanning set is another prerequisite for the definition of a basis.

Definition. A set X is a *spanning set* for V if, for every $v \in V$, there is an n and there are vectors $x_1, \ldots, x_n \in X$ and there are scalars $r_1, \ldots, r_n \in F$ such that

$$v = f_{r_1}^{\mathbb{V}}(x_1) +^{\mathbb{V}} \cdots +^{\mathbb{V}} f_{r_n}^{\mathbb{V}}(x_n).$$

We can now define unordered and ordered bases.

Definition. A set $\mathcal{B} \in V$ is a *basis* for $\mathbb{V}$ if $\mathcal{B}$ is a linearly independent ordered spanning set for $\mathbb{V}$. An ordered set $\mathcal{B} = (b_1, \ldots, b_n) \in V^n$ is an *ordered basis* for $\mathbb{V}$ if $\mathcal{B}$ is a linearly independent ordered spanning set for $\mathbb{V}$.

Proposition. An unordered basis in a superstructure has rank 1.

Proposition. An ordered basis in a superstructure has rank 3.

Monomorphisms

The monomorphism is the connecting link between standard and nonstandard methods. The transfer principle states that a standard sentence holds if and only if its nonstandard form also holds.

Definition. The injection $* : V(X) \to V(Y)$ is called a monomorphism if

(i) ${}^*(\emptyset) = \emptyset$, where $\emptyset$ is the empty set,

(ii) $a \in X$ implies ${}^*a \in Y$, and $n \in N$ implies ${}^*n = n$,

(iii) $a \in V_{n+1}(X) - V_n(X)$ implies ${}^*a \in V_{n+1}(Y) - V_n(Y)$, $n \geq 0$,

(iv) if $a \in {}^*V_n(X)$, $n \geq 1$, and $b \in a$, then $b \in {}^*V_{n-1}(X)$,

(v) *(transfer principle)* for any sentence Φ in $\mathcal{L}_X$, Φ holds in $V(X)$ iff ${}^*\Phi$ holds in $V(Y)$.

The following theorem establishes several fundamental properties of monomorphisms. We include a part of the proof to illustrate the method.

Theorem.

(a) Let $a, b, a_1, \ldots, a_n$ be fixed entities in S. Then

(i) ${}^*\{a_1, \ldots, a_n\} = \{{}^*a_1, \ldots, {}^*a_n\}$

(ii) ${}^*(a_1, \ldots, a_n) = ({}^*a_1, \ldots, {}^*a_n)$

(iii) $a \in b$ iff ${}^*a \in {}^*b$

(iv) $a = b$ iff ${}^*a = {}^*b$

(v) $a \subseteq b$ iff ${}^*a \subseteq {}^*b$

(vi) ${}^*(a \setminus b) = {}^*a \setminus {}^*b$

(vii) ${}^*(\bigcup_{i=1}^n a_i) = \bigcup_{i=1}^n {}^*a_i$, ${}^*(\bigcap_{i=1}^n a_i) = \bigcap_{i=1}^n {}^*a_i$,

(viii) ${}^*(a_1 \times a_2 \times \ldots \times a_n) = {}^*a_1 \times {}^*a_2 \times \ldots \times {}^*a_n$.

(b) If P is a relation on $a_1 \times \ldots \times a_n$ then ${}^*P$ is a relation on ${}^*a_1 \times \ldots \times {}^*a_n$, and, for $n = 2$, ${}^*(\mathrm{dom}\ P) = \mathrm{dom}\ {}^*P$ and ${}^*(\mathrm{range}\ P) = \mathrm{range}\ {}^*P$.

(c) If f is a mapping from a into b then ${}^*f$ is a mapping from ${}^*a$ into ${}^*b$, and ${}^*[f(c)] = {}^*f({}^*c)$ for each $c \in a$. Also f is one-to-one iff ${}^*f$ is one-to-one.

Proof.

(a) (i) Let $b = \{a_1, \ldots, a_n\}$, and let $c = \{{}^*a_1, \ldots, {}^*a_n\}$. Clearly, the sentence $(\forall x \in b)((x = a_1) \vee (x = a_2) \vee \ldots \vee (x = a_n))$ holds. By transforming the sentence, we obtain $(\forall x \in {}^*b)((x = {}^*a_1) \vee (x = {}^*a_2) \vee \ldots \vee (x = {}^*a_n))$. Therefore, ${}^*b \subseteq c$. Furthermore, by transforming the sentences $a_1 \in b, \ldots, a_n \in b$, we obtain ${}^*a_1 \in {}^*b, \ldots, {}^*a_n \in {}^*b$. It follows that $c \subseteq {}^*b$. Since ${}^*b \subseteq c$ and $c \subseteq {}^*b$, we conclude that ${}^*b = c$, and thus ${}^*\{a_1, \ldots, a_n\} = \{{}^*a_1, \ldots, {}^*a_n\}$.

(ii) Let $b = (a_1, \ldots, a_n)$, and let $c = ({}^*a_1, \ldots, {}^*a_n)$. By using the definition of an n-tuple and by applying part (i), we see that

$$\begin{aligned}
{}^*b &= {}^*\{(1, a_1), \ldots, (n, a_n)\} \\
&= \{{}^*(1, a_1), \ldots, {}^*(n, a_n)\} \\
&= \{{}^*\{\{1\}, \{1, a_1\}\}, \ldots, {}^*\{\{n\}, \{a_n\}\}\} \\
&= \{\{{}^*\{1\}, {}^*\{1, a_1\}\}, \ldots, \{{}^*\{n\}, {}^*\{n, a_n\}\}\} \\
&= \{\{\{1\}, \{1, {}^*a_1\}\}, \ldots, \{\{n\}, \{n, {}^*a_n\}\}\} \\
&= \{(1, {}^*a_1), \ldots, (n, {}^*a_n)\} \\
&= ({}^*a_1, \ldots, {}^*a_n).
\end{aligned}$$

(iii) By transforming the sentence $a \in b$, we obtain ${}^*a \in {}^*b$. By the transfer principle, $a \in b \iff {}^*a \in {}^*b$.

(v) The expression $a \subseteq b$ can be written as the sentence $(\forall x \in a)[x \in b]$. By transforming this sentence, we obtain $(\forall x \in {}^*a)[x \in {}^*b]$. This sentence is interpreted as ${}^*a \subseteq {}^*b$. By the transfer principle, $a \subseteq b \iff {}^*a \subseteq {}^*b$. □

References

[1] Hurd, A. E., and P. A. Loeb, *An Introduction to Nonstandard Real Analysis*. Academic Press, Orlando, Florida, 1985.

[2] Burris, S., and H. P. Sankappanavar, *A Course in Universal Algebra*. Springer-Verlag, New York, 1981.